

Happy Holidays from John Sommer

I would like to extend my best wishes for you and your family at this holiday season and thank you for the opportunity to work with you this year to improve your academics.



Tutoring gives me a wonderful opportunity to meet and work with very dedicated students with a keen interest in continuously improving themselves. We live in a highly changing environment. A main goal of mine as your tutor is to teach you how to teach yourself. Learning is a life long journey and can be a most enjoyable one.

When I was growing up I lived in northern Indiana, right in the heart of the Snow Belt. A generous deposit of snow created from moisture of Lake Michigan was very common. Snowflakes are some of the most beautiful works of nature. I encourage you to visit SnowCrystals.com on the Internet. This is a very good site to see beautiful pictures of snowflakes and learn more about the physics and chemistry of snowflakes.

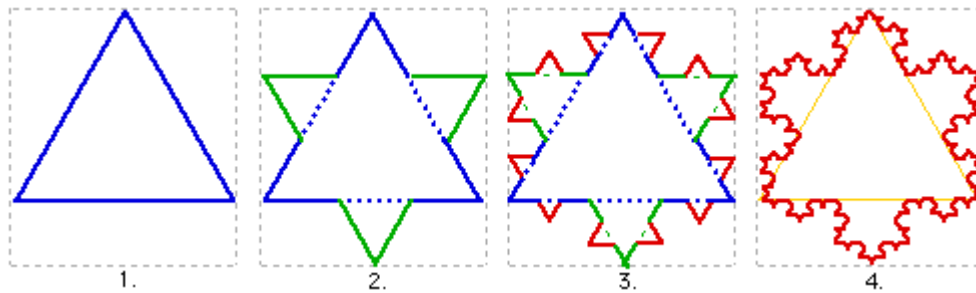
"Nothing in Nature is random... a thing appears random only through the incompleteness of our own knowledge." - Spinoza

Since most of my tutoring is mathematics related, I would like to present to you a mathematical snowflake. The mathematical snowflake is based on "fractals", which is a very new part of mathematics - Fractal Geometry (did not exist when I was your age). The name fractal (from the Latin fractus meaning broken, irregular stone) was coined in 1975 by the mathematician Benoit Mandelbrot to describe geometric shapes that have a similar appearance at any degree of magnification (also called *recursive replacement* and *self-similarity*). There are many objects in nature that are considered to be fractals ranging from mountains and clouds to the microscopic world of our body's vascular system to outer space world of planets and galaxies and to art, movies and music (Star Wars, Toy Story and others use fractals).

The mathematical snowflake I would like to introduce to you is the Koch Snowflake.

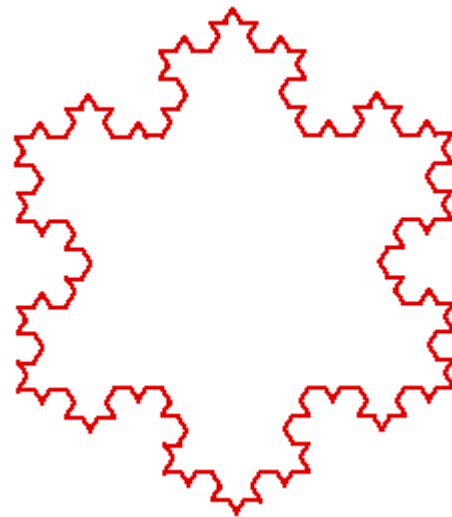
The rules for constructing the Koch Snowflake are very simple.

The Koch Snowflake Curve



1. Start with an equilateral triangle whose sides have length 1.
2. On the middle third of each of the three sides, build an equilateral triangle with sides of length $1/3$. Erase the base of each of the three new triangles. At this point we have the Star of David.
3. On the middle third of each of the twelve sides, build an equilateral triangle with sides of length $1/9$. Erase the base of each of the twelve new triangles.
4. Repeat the process with this 48-sided figure. See the likeness to a crystal of snow emerge?

At the right, step 4 is magnified by a power of two.



The "limit curve" defined by repeating this process an infinite number of times, adding more and more, smaller and smaller triangles at each stage, is called the **Koch's SNOWFLAKE CURVE**, named after Niels Fabian Helge von Koch (Sweden, 1870-1924). Definitely ahead of his time.

The construction of the Koch Snowflake is an example of a *recursive process*, a process in which the same set of rules are applied over and over,

Koch Snowflake Curve

with the end product at each step becoming the starting point for the next step.

Since we are curious, we start to wonder about the properties of this unique geometric figure. Two properties of interest to almost every geometric figure are Perimeter and Area. This figure has some surprising results that may seem paradoxical.

Let's first take a look at the figure from step 4.

- ▶ The snowflake is connected - no gaps or breaks. But it is not smooth, very jagged, like a mountaintop, edges of clouds or an ocean shoreline.
- ▶ The snowflake never escapes the initial square containing our original equilateral triangle. So it encloses a finite area no larger than a credit card. On the other hand, at each step building the new little triangles adds more than one unit of length (actually $(4/3)^{n-1}$ units are added at step n). Thus the length of the snowflake curve is infinitely long.
- ▶ Under a magnifying glass, a little piece of the snowflake looks identical to a larger, unmagnified chunk (*self-similarity*).

The remaining pages look at some of the mathematics behind the Koch Snowflake Curve. If you find fractals interesting, look at <http://a-i-studio.com/snowflake/> for a fractal snowflake generator (picture on page 1 was produced using the program) or more fractal information at <http://spanky.triumf.ca/www/fractint/fractint.html>. You can download software to generate fractals and see some for interesting fractal graphics. You will see some amazing and beautiful fractal pictures. If you want to dive more into the mathematics, look into Sierpinski Gasket, Julies Sets, Pythagorean Trees and Mandelbrot set.

In his classic book *The Fractal Geometry of Nature*, Mandelbrot wrote:
"Why is [standard] geometry often described as "cold" and "dry"? One reason lies in its inability to describe the shape of a cloud, mountain, coastline, or a tree. Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth nor does lightning strike in a straight line... Many patterns of Nature are so irregular and fragmented, that compared with [standard geometry] Nature exhibits not only a higher degree but an altogether different level of complexity. The number of distinct scales of length of natural patterns is for all practical purposes infinite."

Koch Snowflake Curve

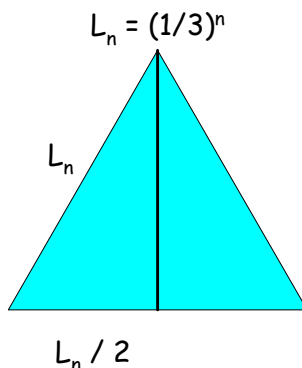
Now let's take a look at some of the mathematics. Since we start with an equilateral triangle of sides one unit, the perimeter is 3 units and the area is $(1/2) \text{ base} * \text{height} = (1/2) * (1) * h_0$ (no need at this time to calculate the height). Let's use Δ to represent the area of our initial triangle.

First let's look at the perimeter.

Step	Number of sides	Perimeter
0	3	3
1	4 * 3 each bump out causes existing side to now be 4 sides so we take 4 * # side previous step sides.	As we bump out we change length of existing sides by 1/3 of previous length so new length of each side is $(1/3) * 1$ perimeter is # sides * length of each $(4 * 3) * (1/3 * 1)$
2	4 * (4 * 3)	Length of sides = $(1/3) * (1/3) * 1$ perimeter = $(1/3)^2 * 4^2 * 3$
3	4 * (4 * 4 * 3)	Length of sides = $(1/3)^3$ Perimeter = $(4/3)^3 * 3$
n	$4^n * 3$	$(4/3)^n * 3$

As you can see, as we add more steps the perimeter keeps increasing - infinitely.

Next we look at area. Each step we change the length of the sides (see previous table) and add new triangles. First we need to find area of each new triangle. See below for the derivation of the formula.



We use pythagorean theorem to get h:

$$h^2 = L_n^2 - (L_n/2)^2$$

$$h^2 = L_n^2 (1 - 1/4)$$

$$h^2 = L_n^2 * h_0^2$$

the initial triangle has its height squared = $(1 - 1/4)$

so we can substitute, then we get

$$h = L_n * h_0$$

so area of new triangle is:

$$A_n = (1/2) * \text{base} * \text{height}$$

$$= (1/2) L_n * (L_n * h_0)$$

$$= L_n^2 * ((1/2) * h_0)$$

$$= L_n^2 * A_0$$

Koch Snowflake Curve

So each new triangle adds $((1/3)^2)^{2n} * \Delta = (1/9)^n * \Delta$

Step	# of new triangles	Area of new triangles	Total Area of new
0			Δ
1	3	$(1/9)^1 * \Delta$	$3 * (1/9)^1 * \Delta$
2	$3 * 4$	$(1/9)^2 * \Delta$	$3 * 4 * (1/9)^2 * \Delta$
3	$3 * 4^2$ one new triangle for each side from previous step (see perimeter table)	$(1/9)^3 * \Delta$	$3 * 4^2 * (1/9)^3 * \Delta$
n	$3 * 4^{(n-1)}$	$(1/9)^n * \Delta$	$3 * 4^{(n-1)} * (1/9)^n * \Delta$

So the total area at any given step will be the area of the previous step plus the area of all of the new triangles. If A_n is the area of the snowflake at step n , then the formula is:

$$A_n = A_{n-1} + 3 * 4^{(n-1)} * (1/9)^n * \Delta \text{ which we can simplify to}$$

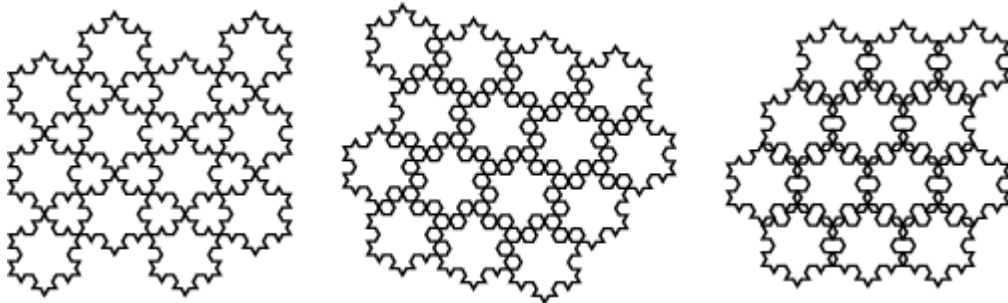
$$A_n = A_{n-1} + (1/3)(4/9)^{(n-1)} * \Delta$$

With a bit more mathematics, we can show that the area

$$A_n = 1 + \frac{1}{3} \sum_{k=0}^{\infty} \left(\frac{4}{9}\right)^k = \left(1 + \left(\frac{1}{3}\right) * \left(\frac{1}{1 - \frac{4}{9}}\right)\right) \Delta = \frac{8}{5} \Delta$$

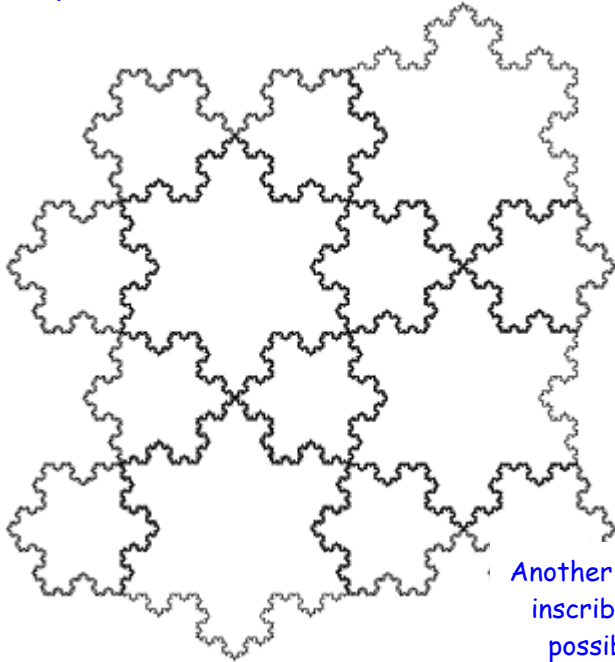
as $n \rightarrow \infty$

Some beautiful tilings, a few examples of which are illustrated below, can be made with iterations of Koch snowflakes.



Koch Snowflake Curve

two sizes of Koch snowflakes in area ratio 1:3 tile the plane, as shown to the left (Mandelbrot).



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Another beautiful modification of the Koch snowflake involves inscribing the constituent triangles with filled-in triangles, possibly rotated at some angle. Some sample results are illustrated above for 3 and 4 iterations.

